Black-box stability analysis of hybrid systems with sample-based multiple Lyapunov functions

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***Abstract*— We present a framework based on multiple Lyapunov functions to find probabilistic data-driven guarantees on the stability of unknown constrained switching linear systems (CSLS), which are switching linear systems whose switching signal is constrained by an automaton. The stability of a CSLS is characterized by its constrained joint spectral radius (CJSR). Inspired by the scenario approach and previous work on unconstrained switching systems, we characterize the number of observations needed to find sufficient conditions on the (in-)stability of a CSLS using the notion of CJSR. More precisely, our contribution is the following: we derive a probabilistic upper bound on the CJSR of an unknown CSLS from a finite number of observations. We also derive a deterministic lower bound on the CJSR. From this we obtain a probabilistic method to characterize the stability of an unknown CSLS.**

I. Introduction

Due to major technological upheavals, the complexity of many dynamical systems has dramatically increased in recent years, thus making their control more and more challenging. The academic community has coined this paradigm shift under the name of the *Cyber-Physical revolution* (see [1]–[5]). In particular, *Hybrid systems*, which often appear in Cyber-Physical applications, are dynamical systems whose dynamics are characterized by continuous and discrete behaviours.

In many practical applications, the engineer cannot rely on having a model, but rather has to analyse the underlying black-box system in a *data-driven* fashion. Most classical data-driven methods (see e.g. [6]–[8]) are limited to linear systems and rely on classical identification and frequency-domain approaches. These methods may not well suited for Cyber-Physical systems because of the natural complexity of these systems. Novel black-box stability analysis methods have been recently developed based on *scenario optimization* (see [9]–[11]). In this paper we seek to take one more step towards complexity.

We consider black-box stability analysis of discrete-time *switching linear systems*. Dynamics of a switching linear system defined by a set of matrices $\mathbf{\Sigma} = \{A_i\}_{i \in \{1,\dots,m\}}$ is given by the following equation:

$$x_{t+1} = A_{\sigma(t)} x_t \tag{1}$$

for any $t \in \mathbb{N}$, where $x_t \in \mathbb{R}^n$ and $\sigma(t) \in \{1, \dots, m\}$ are respectively the *state* and the *mode* at time t. The sequence $(\sigma(0), \sigma(1), \dots) \subseteq \{1, \dots, m\}^{\mathbb{N}}$ is the *switching sequence*.

Switching linear systems are an important family of hybrid systems which often arise in Cyber-Physical systems (see [12]). Stability analysis of switching linear systems is challenging due to the hybrid behaviour caused by the switches. In recent years, many white-box stability analysis techniques have been proposed (see [13], [14] and references therein).

In particular, we are interested in the stability of *constrained switching linear systems* (*CSLS* for short). A CSLS is a switching linear system with logical rules on its switching sequence. We represent these rules by an *automaton* (see Definition 2.1). White-box stability of CSLS has also been studied extensively (see e.g. [15]–[17]). In particular, we are interested in asymptotic stability of CSLS, whose definition is given as follows. Given an automaton $\mathbf{G}$ and a set of matrices $\mathbf{\Sigma}$, the system $S(\mathbf{G}, \mathbf{\Sigma})$ is said to be *asymptotically stable* (or *stable*, for short) if, for all $x \in \mathbb{R}^n$ and for all infinite words $(\sigma(0), \sigma(1), \dots)$ accepted by $\mathbf{G}$,

$$\lim_{t \to \infty} A_{\sigma(t-1)} \dots A_{\sigma(0)} x = 0. \tag{2}$$

In this work we extend the approaches in [9]–[11] by considering a hybrid state space. For a CSLS $S(\mathbf{G}(V, E), \mathbf{\Sigma})$, we consider that one can observe points in $\mathbb{R}^n \times V$ i.e., couples of continuous states and discrete nodes. This allows us to find probabilistic guarantees for the asymptotic stability of CSLS whose dynamics is unknown.

Outline. The rest of this paper is organized as follows. We introduce the problem that we tackle in Section II, as well as all concepts needed to this end. We present our results in Section III. We first propose a black-box formulation allowing us to do this in a data-driven fashion. We then propose a deterministic method to find sufficient condition for instability of black-box CSLS. Finally we find probabilistic guarantees on the stability of a CSLS whose dynamics are unknown. Results are illustrated on a numerical example in Section IV.

Notations. We respectively denote the set of all nonnegative and positive real number by $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$. We also respectively denote by $\mathcal{S}_+^n$ and $\mathcal{S}_{++}^n$ the set of all positive semi-definite and positive definite symmetric matrices. For any matrix $A \in \mathbb{R}^{n \times n}$, we write $A \succ 0$ to state that A is positive definite, and $A \succeq 0$ to state that it is positive semi-definite. We denote by $\|\cdot\|$ the l_2 norm, and, for any matrix $A \in \mathcal{S}^n$, we define $\|\cdot\|_A : \mathbb{R}^n \to \mathbb{R}_{\geq 0} : x \mapsto \sqrt{x^T P x}$ as the

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ellipsoidal norm. Finally, $\mathbb{S}$ denotes the sphere of unit radius centered at the origin.

II. Problem setting

In this section, we introduce the notions necessary to formally write the problem tackled in this paper.

A. Constrained joint spectral radius

We first define an *automaton* (see e.g. [18]):

Definition 2.1: An automaton is a strongly connected[1], directed and labelled graph $\mathbf{G}(V, E)$, where V is the set of nodes and E the set of edges. Note that we drop the writing of V and E when it is clear from the context. The edge $(u, v, \sigma) \in E$ between two nodes $u, v \in V$ carries the label $\sigma \in \{1, \dots, m\}$, where $m \in \mathbb{N}$ is the number of labels.

In the context of CSLS, σ maps to a mode of the system. A sequence of labels $(\sigma(0), \sigma(1), \dots)$ is a *word* in the language *accepted* by the automaton $\mathbf{G}$ if there is a path in $\mathbf{G}$ carrying the sequence as the succession of the labels on its edges. A CSLS defined on the set of matrices $\mathbf{\Sigma}$ and constrained by the automaton $\mathbf{G}$ is noted $S(\mathbf{G}, \mathbf{\Sigma})$.

Let us present an example of CSLS, inspired from [16, Section 4], in order to illustrate the notions defined above.

Example 2.1: Consider a plant that may experience control failures. Its dynamics is given by $x_{t+1} = A_{\sigma(t)} x_t$ where $A_{\sigma(t)} = A + BK_{\sigma(t)}$ with

$$A = \begin{pmatrix} 0.47 & 0.28 \\ 0.07 & 0.23 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3}$$

$K_\sigma(t)$ is described as follows. $K_1 = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$ with $k_1 = -0.245$ and $k_2 = 0.135$, corresponds to the mode where the controller works as expected. $K_2 = \begin{pmatrix} 0 & k_2 \end{pmatrix}$ and $K_3 = \begin{pmatrix} k_1 & 0 \end{pmatrix}$ respectively correspond to the modes when the first and the second part of the controller fails. And $K_4 = \begin{pmatrix} 0 & 0 \end{pmatrix}$ corresponds to the mode when both parts fail. We consider as a constraint that the same part of the controller never fails twice in a row. This is modelled by the automaton $\mathbf{G}$, depicted in Figure 2.1.

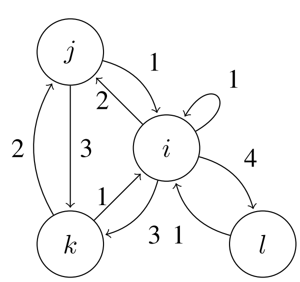

Fig. 1. Automaton $\mathbf{G}$. No mode can fail twice in a row.

In this example, the considered CSLS is thus $S(\mathbf{G}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} = \{A_1, A_2, A_3, A_4\}$.

[1]A strongly connected graph is a graph that has a path from each vertex to every other vertex. See [18, Definition 2.2.13] for a formal definition.

The *constrained joint spectral radius*, introduced in [15], is defined as follows:

Definition 2.2 ([15, Definition 1.2]): Given a set of matrices $\mathbf{\Sigma} = \{A_1, \dots, A_m\}$ and an automaton $\mathbf{G}$ whose labels are $\sigma \in \{1, \dots, m\}$, the *constrained joint spectral radius* (*CJSR* for short) of the CSLS $S(\mathbf{G}, \mathbf{\Sigma})$ is defined as

$$\rho(\mathbf{G}, \mathbf{\Sigma}) = \lim_{t \to \infty} \max\{\|A_{\sigma(t-1)} \dots A_{\sigma(0)}\|^{1/t} : (\sigma(0), \dots, \sigma(t-1)) \text{ is a word of } \mathbf{G}\}. \tag{4}$$

As the following proposition shows, the CJSR characterizes the stability of a CSLS:

Proposition 2.1 ([15, Corollary 2.8]): Given a set of matrices $\mathbf{\Sigma}$ and an automaton $\mathbf{G}$, the CSLS $S(\mathbf{G}, \mathbf{\Sigma})$ is asymptotically stable if and only if $\rho(\mathbf{G}, \mathbf{\Sigma}) < 1$.

B. Multiple Quadratic Lyapunov Functions

We present a classical result from model-based analysis of CSLS. The following proposition gives a quadratic framework for approximating the CJSR of a given CSLS:

Proposition 2.2 ([19, Proposition 2.20]): Consider a CSLS $S(\mathbf{G}(V, E), \mathbf{\Sigma})$ and a constant $\gamma > 0$. If there exists a set of quadratic forms $\{P_i,\, i \in V\} \subset \mathcal{S}_+^n$ satisfying the set of *Linear Matrix Inequalities* (*LMI*s)

$$\forall (u, v, \sigma) \in E : A_\sigma^T P_v A_\sigma \preceq \gamma^2 P_u, \tag{5}$$

then $n^{-1/2}\gamma \leq \rho(\mathbf{G}, \mathbf{\Sigma}) \leq \gamma$.

If $\gamma < 1$, the set of norms $\{\|\cdot\|_{P_u},\, u \in V\}$ is called a set of *Multiple Quadratic Lyapunov Functions* (*MQLF*). Proposition 2.2 thus gives a sufficient condition for the stability of a given CSLS using MQLF.

Consider a given CSLS $S(\mathbf{G}(V, E), \mathbf{\Sigma})$. Let $\Delta = \mathbb{S} \times E$. As a preparation to develop our data-driven approach, we reformulate the stability condition in Proposition 2.2 into a robust optimization problem[2]:

$$\mathcal{P}(\Delta) : \min_{\substack{\{P_u,\, u \in V\} \subset \mathcal{S}_{++}^n \\ \gamma \geq 0}} \gamma \tag{6a}$$

$$\text{s.t. } \forall (x, (u, v, \sigma)) \in \Delta : (A_\sigma x)^T P_v (A_\sigma x) \leq \gamma^2 x^T P_u x \tag{6b}$$

We denote by $\gamma^*(\Delta)$ and $\{P_u^*(\Delta),\, u \in V\}$ the solution of $\mathcal{P}(\Delta)$. Following Proposition 2.2, if $\gamma^*(\Delta) < 1$, the set $\{P_u^*(\Delta),\, u \in V\}$ is a set of MQLF.

The notation $\mathcal{P}(\Delta)$ emphasizes that the whole set of constraints is known in this white-box formulation, in opposition to the white-box problem $\mathcal{P}(\omega_N)$ defined in (8).

[2]Note that we can restrict x to the unit sphere $\mathbb{S}$ in constraint (6b). We can do this thanks to the *homogeneity* of the CSLS: for any $x \in \mathbb{R}^n$, $\mu > 0$, and $A \in \mathbf{\Sigma}$, it holds that $A(\mu x) = \mu A x$.

III. MAIN RESULTS

A. White-box formulation

In this paper, we analyze the problem of approximating the CJSR in a data-driven fashion: we assume that the system is not known, hence problem $\mathcal{P}(\Delta)$ defined in Equation (6) cannot be solved. We only sample a finite number N of observations of a given CSLS $S(\mathbf{G}(V,E),\mathbf{\Sigma})$. One observation consists of an ordered pair of points in the state space defined above i.e., $\mathbb{R}^n \times V$. The i-th observation is a couple of initial and final points in $\mathbb{R}^n$ and nodes in V. It is noted $((x_i,u_i),(y_i,v_i)) \in (\mathbb{R}^n \times V)^2$ where $(u_i,v_i,\sigma_i) \in E$ for some label $\sigma_i \in \{1,\dots,m\}$, and $y_i = A_{\sigma_i}x_i$. For any $i = 1,\dots,N$, x_i and (u_i,v_i,σ_i) are drawn randomly, uniformly and independently from respectively $\mathbb{S}$ and E. We attract the attention of the reader on the fact that the sampled mode is not known.

Mathematically, the sample set ω_N is defined from the observations as

$$\omega_N = \{(x_i,(u_i,v_i,\sigma_i),\, i = 1,\dots,N\}, \tag{7}$$

where x_i, u_i, v_i and σ_i are as described above. Note that ω_N is a subset of N elements of Δ.

Now, for a given set ω_N, let us define the *sampled optimization problem* $\mathcal{P}(\omega_N)$:

$$\mathcal{P}(\omega_N): \min_{\substack{\{P_u,\, u\in V\}\subset\mathcal{S}^n_{++}\\ \gamma\geq 0}} \gamma \tag{8a}$$

$$\text{s.t. } \forall(x,(u,v,\sigma)) \in \omega_N : (A_\sigma x)^T P_v (A_\sigma x) \leq \gamma^2 x^T P_u x \tag{8b}$$

$$\forall u \in V : P_u \in \{P : I \preceq P \preceq CI\}, \tag{8c}$$

for a large $C \in \mathbb{R}_{\geq 0}$. We denote by $\gamma^*(\omega_N)$ and $\{P^*_u(\omega_N),\, u \in V\}$ the solution of $\mathcal{P}(\omega_N)$. The problem that we tackle in this paper is the inference, with a user-defined confidence level, of $\gamma^*(\Delta)$, the solution of $\mathcal{P}(\Delta)$ defined in Equation (6) from the solution of $\mathcal{P}(\omega_N)$ defined in Equation (8) i.e., the value $\gamma^*(\omega_N)$ and the set $\{P^*_u(\omega_N),\, u \in V\}$.

Problem $\mathcal{P}(\omega_N)$ defined in Equation (8) differs from $\mathcal{P}(\Delta)$ defined in Equation (6) in two ways: the LMIs expressed in constraint (8b) are restricted to ω_N, and compactness of the domain of the matrices $\{P_u, u \in V\} \subset \mathcal{S}^n_{++}$ is imposed in constraint (8c). We will need the latter to prove Proposition 3.2.

B. Deterministic lower bound on the CJSR

In the same fashion as in [9], we derive a deterministic lower bound on the CJSR:

Proposition 3.1: Let ω_N be a set of N observations from Δ as explained above. Consider the program $\mathcal{P}(\omega_N)$ defined in (8) for the CSLS $S(\mathbf{G},\mathbf{\Sigma})$ with optimal cost $\gamma^*(\omega_N)$. Then the following holds :

$$n^{-1/2}\gamma^*(\omega_N) \leq \rho(\mathbf{G},\mathbf{\Sigma}). \tag{9}$$

Proof: Notice that $\mathcal{P}(\omega_N)$ defined in (8) is a relaxation of $\mathcal{P}(\Delta)$ defined in (6). As a consequence, we have $\gamma^*(\Delta) \geq \gamma^*(\omega_N)$. Following Proposition 2.2,

$$\rho(\mathbf{G},\mathbf{\Sigma}) \geq n^{-1/2}\gamma^*(\Delta) \geq n^{-1/2}\gamma^*(\omega_N), \tag{10}$$

which is the desired result. ∎

Remark 3.1: One can show that the lower bound of Proposition 3.1 can be improved thanks to *Sums-of-Squares approximation methods*, introduced in [20] for the approximation of the *joint spectral radius* and generalized in [16] for the CJSR.

C. Probabilistic upper bound on the CJSR

Proposition 3.2: Consider the program $\mathcal{P}(\Delta)$ for the CSLS $S(\mathbf{G}(V,E),\mathbf{\Sigma})$ with optimal cost $\gamma^*(\Delta)$. There exists a set $\omega \subset \Delta$ with $|\omega| = |V|n(n+1)/2$ such that $\gamma^*(\omega) = \gamma^*(\Delta)$, where $\gamma^*(\omega)$ is the optimal cost of the program $\mathcal{P}(\omega)$.

Proof: First, from the arguments in [11, Lemma 1], we claim that there exists $\omega \subset \Delta$ with $|\omega| = |V|n(n+1)/2+1$ such that $\gamma^*(\omega) = \gamma^*(\Delta)$. Now, we consider the problem $\mathcal{P}(\omega)$ as defined in (8). With a similar argument as the one in [10, Theorem 2], we can conclude that the objective remains unchanged removing one of the points in ω. ∎

Remark 3.2: There are two main differences between Proposition 3.2 and [11, Lemma 1]: the proposition is derived for CSLS instead of arbitrary switching linear systems, and the cardinality of the set is the number of variables of the program minus 1, while it is the number of variables of the program in [11].

Now, let us define the notion of *spherical cap*:

Definition 3.1 ([21]): The *spherical cap* on $\mathbb{S}$, the unit sphere, of direction c and measure ε is defined as $\mathcal{C}(c,\varepsilon) := \left\{x \in \mathbb{S} : c^T x > \|c\|\delta(\varepsilon)\right\}$, where $\delta(\varepsilon)$ is defined as[3]

$$\delta(\varepsilon) = \sqrt{1 - I^{-1}\left(2\varepsilon; (n-1)/2, 1/2\right)}. \tag{12}$$

The following proposition provides a bound on the conservatism of the sampled problem $\mathcal{P}(\omega_N)$ defined in (8), with respect to the white-box problem $\mathcal{P}(\Delta)$ defined in (6) as a function of N, the number of points sampled:

Proposition 3.3: Consider the program $\mathcal{P}(\Delta)$ for the CSLS $S(\mathbf{G}(V,E),\mathbf{\Sigma})$ with optimal cost $\gamma^*(\Delta)$, and where $\mathbf{\Sigma}$ contains m modes. Let $\omega_N = \{(x_i,(u_i,v_i,\sigma_i)), i = 1,\dots,N\}$ be a set of N samples from Δ as explained above. Suppose $N \geq |V|n(n+1)/2$. Then, for all $\varepsilon \in (0,1]$, with probability at least

$$\beta(\varepsilon,m,N) = 1 - |V|\frac{n(n+1)}{2}\left(1 - \frac{\varepsilon}{m|V|}\right)^N, \tag{13}$$

[3] In Equation (12), $I^{-1}(y;a,b)$ is the *inversed regularized incomplete beta function* (see [22]). Its ouput is $x > 0$ such that $I(x;a,b) = y$, where I is defined as

$$I(\cdot;a,b) : \mathbb{R}_{>0} \to \mathbb{R}_{>0} : x \mapsto I(x;a,b) = \frac{\int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t}{\int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t} \tag{11}$$

there exists a set $\omega'_N = \{(x'_i, (u_i, v_i, \sigma_i)), i = 1, \ldots, N\} \subset \Delta$ such that $\gamma^*(\omega'_N) = \gamma^*(\Delta)$ with $\|x_i - x'_i\| \leq \sqrt{2 - 2\delta(\varepsilon)}$.

Proposition 3.3 is an extension of [11, Proposition 2] to the constrained case. The two propositions differ in three different ways. First the number of variables of the problem is not the same. Second, given that the edges are sampled uniformly (see Section II), the probability of sampling a certain label σ is at least $1/(m|V|)$, while it is $1/m$ in the unconstrained case. Third, Proposition 3.2 allows to improve the probability β according to Remark 3.2.

We now apply a sensitivity analysis approach in order to obtain from Proposition 3.3 a probabilistic upper bound on $\gamma^*(\Delta)$ the optimal cost of $\mathcal{P}(\Delta)$ (defined in Equation (6)) from the sampled optimal variables $\gamma^*\omega_N$ and $\{P^*_u(\omega_N), u \in V\}$ of $P(\omega_N)$ (defined in Equation (8)).

Theorem 3.4: Consider the program $\mathcal{P}(\Delta)$ defined in (6) for the CSLS $S(\mathbf{G}(V,E), \mathbf{\Sigma})$ with optimal cost $\gamma^*(\Delta)$. Let ω_N be a set of N samples from Δ as explained in Section III-A, with $N \geq |V|n(n+1)/2$. Consider the sampled program $\mathcal{P}(\omega_N)$ defined in (8) with solution $\gamma^*(\omega_N)$ and $\{P^*_u(\omega_N), u \in V\}$. For any $\beta \in [0, 1)$, let

$$\varepsilon = m|V| \left(1 - \sqrt[N]{\frac{2(1-\beta)}{|V|n(n+1)}}\right). \tag{14}$$

Then, with probability at least β,

$$\gamma^*(\Delta) \leq \gamma^*(\omega_N) + \max_{(x,(u,v,\sigma)) \in \omega_N} \left\{ \sqrt{\frac{\lambda^u_{\max}}{\lambda^u_{\min}}} \gamma^*(\omega_N) + \sqrt{\frac{\lambda^v_{\max}}{\lambda^u_{\min}}} \mathcal{A}(\mathbf{\Sigma}) \right\} d(\varepsilon), \tag{15}$$

with $d(\varepsilon) = \sqrt{2 - 2\delta(\varepsilon)}$, $\lambda^u_{\min}$ and $\lambda^u_{\max}$ respectively the minimal and maximal eigenvalue of $P^*_u(\omega_N)$, and

$$\mathcal{A}(\mathbf{\Sigma}) = \max_{A \in \mathbf{\Sigma}} \|A\|. \tag{16}$$

Proof: For the sake of readibility, let $\gamma = \gamma^*(\omega_N)$ and $P_u = P^*_u(\omega_N)$ for any $u \in V$. By definition, for any $(x, (u, v, \sigma)) \in \omega_N$,

$$\|A_\sigma x\|_{P_v} \leq \gamma \|x\|_{P_u}. \tag{17}$$

Consider now for any $P \in \mathcal{S}^n$ its *Cholesky decomposition* $P = L^T L$, where $\mathcal{S}^n$ is the set of *positive semi-definite symmetric* matrices. Then the following holds:

$$\|x\|_P = \|Lx\| \leq \|L\|\|x\| \leq \sqrt{\lambda_{\max}(P)}\|x\|, \tag{18}$$

where $\lambda_{\max}(P)$ is the maximal eigenvalue of P. Let us now consider an arbitrary constraint $(y, (u, v, \sigma)) \in \Delta$, and define $y = x + \Delta x$ with $(x, (u, v, \sigma)) \in \omega_N$. Then, for any $(x, (u, v, \sigma)) \in \omega_N$, following inequalities (17) and (18), and using triangle and Cauchy–Schwarz inequalities, it holds that

$$\begin{aligned}
\|A_\sigma(x + \Delta x)\|_{P_v} &\leq \|A_\sigma x\|_{P_v} + \|A_\sigma \Delta x\|_{P_v} \\
&\leq \gamma \|x\|_{P_u} + \|A_\sigma \Delta x\|_{P_v} \\
&= \gamma \|(x + \Delta x) - \Delta x\|_{P_u} + \|A_\sigma \Delta x\|_{P_v} \\
&\leq \gamma \|x + \Delta x\|_{P_u} + \gamma \|\Delta x\|_{P_u} + \|A_\sigma \Delta x\|_{P_v} \\
&\leq \gamma \|x + \Delta x\|_{P_u} + \gamma \|\Delta x\| \sqrt{\lambda^u_{\max}} \\
&\qquad + \|A_\sigma\| \|\Delta x\| \sqrt{\lambda^v_{\max}} \\
&\leq \gamma \|x + \Delta x\|_{P_u} + \gamma \|\Delta x\| \sqrt{\lambda^u_{\max}} \frac{\|x + \Delta x\|_{P_u}}{\sqrt{\lambda^u_{\min}}} \\
&\qquad + \|A_\sigma\| \|\Delta x\| \sqrt{\lambda^v_{\max}} \frac{\|x + \Delta x\|_{P_u}}{\sqrt{\lambda^u_{\min}}} \\
&= \left[\gamma + \left(\sqrt{\frac{\lambda^u_{\max}}{\lambda^u_{\min}}} \gamma + \sqrt{\frac{\lambda^v_{\max}}{\lambda^u_{\min}}} \|A_\sigma\|\right) \|\Delta x\|\right] \\
&\qquad \|x + \Delta x\|_{P_u}.
\end{aligned} \tag{19}$$

For any $\beta \in [0, 1)$, let ε be defined such as in Equation (14), then, given that $N \geq |V|n(n+1)/2$, Proposition 3.3 guarantees the existence of a set ω'_N with N points such that $\gamma^*(\omega'_N) = \gamma^*(\Delta)$ with probability at least β, and such that for any $(x, (u, v, \sigma)) \in \omega_N$, there exists Δx such that $(x + \Delta x, (u, v, \sigma)) \in \omega'_N$ and $\|\Delta x\| \leq d(\varepsilon)$. Hence, by definition and following Equation (19),

$$\begin{aligned}
\gamma^*(\Delta) &= \gamma^*(\omega'_N) \\
&\leq \gamma + \\
&\max_{(x,(u,v,\sigma)) \in \omega_N} \left\{ \sqrt{\frac{\lambda^u_{\max}}{\lambda^u_{\min}}} \gamma + \sqrt{\frac{\lambda^v_{\max}}{\lambda^u_{\min}}} \mathcal{A}(\mathbf{\Sigma}) \right\} d(\varepsilon),
\end{aligned} \tag{20}$$

with probability at least β. ■

D. Estimation of the maximal norm

In order to get a fully data-driven probabilistic bound as expressed in Equation (20), it remains to approximate $\mathcal{A}(\mathbf{\Sigma})$ as defined in Equation (16). First, note that the following holds [14, Proposition 2.7]:

$$\begin{aligned}
\mathcal{A}(\mathbf{\Sigma}) &= \eta^*(\Delta) \\
&= \min_{\eta \geq 0} \eta \text{ s.t. } \forall (x, (u, v, \sigma)) \in \Delta : \|A_\sigma x\| \leq \eta.
\end{aligned} \tag{21}$$

As it is assumed that $\mathbf{\Sigma}$ is not known, in this subsection, we seek to find a probabilistic upper bound on the value of $\mathcal{A}(\mathbf{\Sigma})$, from the given set of observations ω_N. With the same idea as in Section III-C, let us infer the value of $\eta^*(\Delta) = \mathcal{A}(\mathbf{\Sigma})$ from the solution of its sampled problem

$$\eta^*(\omega_N) = \min_{\eta \geq 0} \eta \text{ s.t. } \forall (x, (u, v, \sigma)) \in \omega_N : \|A_\sigma x\| \leq \eta, \tag{22}$$

with a user-defined confidence level.

The general *chance-constrained* theorem [10, Theorem 6] requires a technical assumption [10, Assumption 8] that can be violated in our case. We give a proof for Theorem 3.5 allowing us to get rid of this assumption.

Theorem 3.5: Let ω_N be a set of N samples from Δ as explained in Section III-A. Consider the solutions $\eta^*(\Delta)$ and $\eta^*(\omega_N)$ defined in equations (21) and (22) respectively. For any $\beta' \in [0, 1)$, let

$$\varepsilon' = 1 - \sqrt[N]{1 - \beta'}. \tag{23}$$

Then, with probability at least β',

$$\eta^*(\Delta) \leq \frac{\eta^*(\omega_N)}{\delta(\varepsilon' m|V|/2)}. \tag{24}$$

Proof: Let the *violating set* $V(\eta) := \{(x, (u, v, \sigma)) \in \Delta : \|A_\sigma x\| > \eta\}$, and let $f : \mathbb{R} \to [0, 1] : \eta \mapsto f(\eta) = \mathbb{P}[V(\eta)]$ be its measure. Note that f is decreasing. For any $\varepsilon' \in [0, 1]$, we start by showing the following equation:

$$\mathbb{P}^N[\omega_N \subset \Delta : f(\eta^*(\omega_N)) \leq \varepsilon'] = 1 - (1 - \varepsilon')^N. \tag{25}$$

Consider one sampled constraint $d \in \Delta$, and let $\eta_{\varepsilon'} \in \mathbb{R}$ be such that $f(\eta_{\varepsilon'}) = \varepsilon'$. Then $\mathbb{P}[d \in \Delta : f(\eta^*(\{d\})) > \varepsilon'] = \mathbb{P}[d \in \Delta : f(\eta^*(\{d\})) > f(\eta_{\varepsilon'})]$. Since f is decreasing and has $[0, 1]$ as codomain, $\mathbb{P}[d \in \Delta : f(\eta^*(\{d\})) > f(\eta_{\varepsilon'})] = 1 - \varepsilon'$, hence the following holds:

$$\mathbb{P}[d \in \Delta : f(\eta^*(\{d\})) > \varepsilon'] = 1 - \varepsilon'. \tag{26}$$

Since samples in ω_N are i.i.d., the following holds:

$$\begin{aligned} &\mathbb{P}^N[\omega_N \subset \Delta : f(\eta^*(\omega_N)) > \varepsilon'] \\ &= (\mathbb{P}[d \in \Delta : f(\eta^*(\{d\})) > \varepsilon'])^N \\ &= (1 - \varepsilon')^N, \end{aligned} \tag{27}$$

which is equivalent to Equation (25).

Now, define the projected violating set $\tilde{\mathbb{S}} \subseteq \mathbb{S}$ as follows:

$$\tilde{\mathbb{S}} = \{x \in \mathbb{S} : \exists (u, v, \sigma) \in E, \|A_\sigma x\| > \eta^*(\omega_N)\}. \tag{28}$$

For any $(u, v, \sigma) \in E$, we define:

$$\tilde{\mathbb{S}}_{(u,v,\sigma)} = \{x \in \mathbb{S} : \|A_\sigma x\| > \eta^*(\omega_N)\}. \tag{29}$$

Thus, $\tilde{\mathbb{S}} = \cup_{(u,v,\sigma) \in E} \tilde{\mathbb{S}}_{(u,v,\sigma)}$. In the worst case, the sets $\{\tilde{\mathbb{S}}_{(u,v,\sigma)}\}$ are disjoint. In this case, $\mathbb{P}_x[\tilde{\mathbb{S}}] = \sum_{(u,v,\sigma)\in E} \mathbb{P}_x[\tilde{\mathbb{S}}_{(u,v,\sigma)}]$ and

$$\begin{aligned} \mathbb{P}[V(\eta)] &= \sum_{(u,v,\sigma)\in E} \mathbb{P}_x[\tilde{\mathbb{S}}_{(u,v,\sigma)}] \mathbb{P}_E[\{(u, v, \sigma)\}] \\ &\geq \frac{1}{m|V|} \sum_{(u,v,\sigma)\in E} \mathbb{P}_x[\tilde{\mathbb{S}}_{(u,v,\sigma)}] = \frac{\mathbb{P}_x[\tilde{\mathbb{S}}]}{m|V|}, \end{aligned} \tag{30}$$

where $\mathbb{P}_x$ and $\mathbb{P}_E$ denote the uniform (probability) measure on $\mathbb{S}$ and E respectively. This means that $\mathbb{P}[V(\eta)] \leq \varepsilon'$ implies $\mathbb{P}_x[\tilde{\mathbb{S}}] \leq \varepsilon' m|V|$.

Because of space constraints, we refer the reader to the proof of [9, Theorem 15] for the end of this proof as it follows similar lines. ∎

Theorem 3.5 allows us to directly derive the following corollary:

Corollary 3.1: Consider the program $\mathcal{P}(\Delta)$ defined in (6) for the CSLS $S(\mathbf{G}(V, E), \mathbf{\Sigma})$ with optimal cost $\gamma^*(\Delta)$. Let ω_N be a set of N samples from Δ as explained in Section III-A, with $N \geq |V|n(n + 1)/2$. Consider the sampled program $\mathcal{P}(\omega_N)$ defined in (8) with solution $\gamma^*(\omega_N)$ and $\{P_u^*(\omega_N), u \in V\}$. For any $\beta, \beta' \in [0, 1)$, let

$$\varepsilon = m|V| \left(1 - \sqrt[N]{\frac{2(1 - \beta)}{|V|n(n + 1)}}\right), \tag{31}$$

and

$$\varepsilon' = \frac{m}{2}\left(1 - \sqrt[N]{1 - \beta'}\right). \tag{32}$$

Then, with probability at least $\beta + \beta' - 1$,

$$\rho(\mathbf{G}, \mathbf{\Sigma}) \leq \gamma^*(\omega_N) + \max_{(x,(u,v,\sigma))\in\omega_N} \left\{\sqrt{\frac{\lambda^u_{\max}}{\lambda^u_{\min}}}\gamma^*(\omega_N) + \sqrt{\frac{\lambda^v_{\max}}{\lambda^u_{\min}}}\frac{\eta^*(\omega_N)}{\delta(\varepsilon')}\right\} d(\varepsilon), \tag{33}$$

with $d(\varepsilon) = \sqrt{2 - 2\delta(\varepsilon)}$, $\lambda^u_{\min}$ and $\lambda^u_{\max}$ respectively the minimal and maximal eigenvalue of $P_u^*(\omega_N)$

Proof: Following Proposition 2.2, Equation (33) holds if Equation (15) and Equation (24) both hold. Theorem 3.4 states that Equation (15) holds with probability β, and Theorem 3.5 states that Equation (24) holds with probability β'. Thus

$$\begin{aligned} &\mathbb{P}^N[\omega_N \subset \Delta : \text{(15) and (24) hold}] \\ &= 1 - \mathbb{P}^N[\omega_N \subset \Delta : \text{(15) or (24) does not hold}] \\ &\geq 1 - \mathbb{P}^N[\omega_N \subset \Delta : \text{(15) does not hold}] \\ &\qquad - \mathbb{P}^N[\omega_N \subset \Delta : \text{(24) does not hold}] \\ &\geq 1 - (1 - \beta) - (1 - \beta') \\ &= \beta + \beta' - 1, \end{aligned} \tag{34}$$

which concludes the proof. ∎

IV. Numerical experiments

Let us consider the CSLS $S(\mathbf{G}, \mathbf{\Sigma})$ introduced in Example 2.1. Using the CJSR white-box approximation method introduced in [16], we know that the true CJSR $\rho(\mathbf{G}, \mathbf{\Sigma}) \approx 0.48741$.

The simulations are the following: for different values of N, we sample N observations as explained in Section III-A. We then compute the optimal variables $\gamma^*(\omega_N)$ and $\{P_u^*(\omega_N), u \in V\}$ of the problem $\mathcal{P}(\omega_N)$ defined in Equation (8). From these variables, we compute the lower and upper bounds expressed in Proposition 3.1 and Corollary 3.1. We provide the results for the example described above in Figure IV for an increasing number N of sampled points i.e. $N \in [1, 50000]$.

We observe that the lower bound fastly converges to a conservative value. We recall though that this lower bound is deterministic. Concerning the upper bounds, we notice that an upper bound becomes tighter for larger values N, the number of samples. We also observe that, as expected, the cost of a tighter bound is a smaller confidence level. Indeed, one can see on Figure IV that the bound is tighter for small values of $\beta + \beta' - 1$. We can finally observe

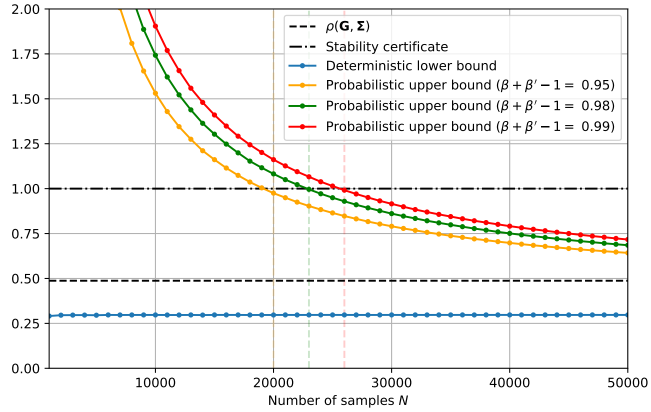

Fig. 2. Lower and upper bounds derived in Proposition 3.1 and Corollary 3.1 for an increasing number of samples N, with confidence levels $\beta + \beta' - 1 \in \{0.95, 0.98, 0.99\}$.

that one needs less samples to have stability guarantee (according to Proposition 2.1), for smaller confidence levels. One needs respectively 20000, 23000 and 26000 samples to have stability guarantee for the considered CSLS with confidence levels of respectively 95%, 98% and 99%.

V. Conclusion

In this work, we leveraged approaches such as *scenario optimization* and *sensitivity analysis* to propose a method providing probabilistic guarantees on the stability of an unknown CSLS. We used the CJSR as a tool to approximate the black-box stability of CSLS. In particular, we provided a deterministic lower bound on the CJSR, as well as a probabilistic upper bound on it. We showed that we obtain tighter approximations of the CJSR for a large number of samples, but also for smaller confidence levels. Finally, we demonstrated that the theory holds by applying it to an academic example.

Our work, and our findings, follow the previous work of [9]–[11]. Compared with this previous body of work, we believe that our contribution achieves an important step towards practical applications, and in particular towards hybrid automata and cyber-physical systems. In the future, we plan to pursue further this direction, for instance by considering more involved models of hybrid systems, and by refining our bounds.

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